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Nonlinear wave interactions in shear flows. Part 2. Third-order theory

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The temporal evolution of a resonant triad of wave components in a parallel shear flow has been investigated at second order in the wave amplitudes by Craik (1971) and Usher & Craik (1974). The present work extends these analyses to include terms of third order and thereby develops the resonance theory to the same order of approximation as the non-resonant third-order theory of Stuart (1960, 1962).

Asymptotic analysis for large Reynolds numbers reveals that the magnitudes of the third-order interaction coefficients, like certain of those at second order, are remarkably large. The implications of this are discussed with particular reference to the roles of resonance and of three-dimensionality in nonlinear instability and to the range of validity of the perturbation analysis.

1. Introduction

The pioneering work of Stuart (1960) on the nonlinear stability of parallel flows concerned the temporal evolution of a single wave component, including terms of third order in the wave amplitude. The third-order terms derive from the interaction of the wave with both its second harmonic and the second-order modification which it induces in the primary flow. The equation governing the complex wave amplitude $A(t)$ is of the now familiar form

$$dA/dt = \alpha c_I A + \lambda |A|^2 A, \tag{1.1}$$

where t denotes time, αc_I is the (real) exponential growth or decay rate predicted by linear theory and λ is the Landau constant, which is generally complex. It is well known that the effect of the nonlinear term depends crucially on the sign of the real part λ_R of λ. Numerical calculations of λ for plane Poiseuille flow and plane Couette–Poiseuille flow have been carried out by Reynolds & Potter (1967) and by Pekeris & Shkoller (1967, 1969), who found that, sufficiently near the critical Reynolds number R_c of linear theory, λ_R is positive, indicating that these flows exhibit finite amplitude subcritical instability.

Corresponding analyses of spatial evolution (Watson 1962) and of the spatial–temporal evolution of a localized disturbance (in the series of papers by Hocking, Stewartson and Stuart 1971–72) also reveal the importance of the sign of λ_R. At Reynolds numbers slightly greater than R_c the analyses of Stewartson & Stuart (1971), Hocking, Stewartson & Stuart (1972) and Davey, Hocking & Stewartson (1974) together yield a correct description of the development of sufficiently small but otherwise arbitrary localized disturbances, since an initial period of development governed by linear theory results in the emergence of a single dominant (but modulated) wave mode. However, for larger initial disturbances, the period of linear development may be insufficient to suppress all other wave modes, and (1.1) or its spatial–temporal counterpart need not apply. Also, for $R < R_c$ one cannot invoke linear theory as a means of singling out the least damped mode and *then* consider its nonlinear evolution. Accordingly, a 'dominant-mode' nonlinear theory for $R < R_c$ relates only to a rather restricted type of initial disturbance which is itself dominated by a single wave component.

Stuart (1962), Benney & Lin (1960) and Benney (1961, 1964) have examined cases where two wave modes interact at third order. These two modes comprise a two-dimensional plane wave and a three-dimensional wave with the same streamwise wavenumber and with a prescribed spanwise periodicity (the latter wave may be thought of as the sum of two identical oblique plane waves propagating at equal and opposite angles to the stream direction). Their interaction at third order yields two coupled equations for the evolution of their amplitudes, containing four third-order interaction coefficients instead of the single Landau constant of (1.1); see Stuart [1962, equations (4.1) and (4.2)]. Examination of this particular form of disturbance was largely prompted by the experimental work of Klebanoff & Tidstrom (1959) and Klebanoff, Tidstrom & Sargent (1962) on the growth of three-dimensionality in unstable boundary layers.

In an attempt to explain the development of subharmonic disturbances in unstable jets and shear layers, Kelly (1968) examined resonance among two-dimensional waves in several inviscid shear flows. Also, Raetz (1959) first showed that Tollmien–Schlichting waves might interact resonantly in boundary layers.

Three-dimensional resonant interactions in shear flows were examined by Craik (1971) and Usher & Craik (1974), hereafter denoted by I and II for brevity, and it is from these papers that the present work has developed. They concern the evolution of a dominant triad of plane waves which interact resonantly at second order. The type of triad chosen comprises a downstream-propagating plane wave and two oblique plane waves propagating at equal and opposite angles to the flow direction. Because of the resonance condition, the downstream wavenumber of these oblique waves is just *half* that of the two-dimensional wave. It is shown in I that triads of this form are likely to exist for a large class of shear flows and particular examples are given for the Blasius boundary layer and a piecewise-linear boundary-layer profile.

To confine attention to triads of this special form is less restrictive than might at first appear. For, as shown in I, the components of such triads interact particularly strongly at large Reynolds numbers, owing to a powerful nonlinear mechanism in the vicinity of the critical layer, where the downstream phase velocity of the waves equals the velocity of the primary flow (for such triads, the respective critical layers of the three waves necessarily coincide at resonance; for any other resonant triad, the critical layers are normally distinct and the non-linear critical-layer mechanism is less powerful). This mechanism may transfer additional energy from the mean flow to the waves, and particularly favours the growth of the oblique-wave components. Accordingly, it may play a significant part in the spontaneous development of three-dimensionality in unstable flows.

The interaction equations for the temporal evolution of the respective complex wave amplitudes A_i $(i = 1, 2, 3)$ are of the form [see II, equations (4.6)]

$$\left.\begin{aligned} dA_1/dt &= \tfrac{1}{2}\alpha c_I A_1 + a_1 A_3 A_2^*, \\ dA_2/dt &= \tfrac{1}{2}\alpha c_I A_2 + a_2 A_3 A_1^*, \\ dA_3/dt &= \alpha \tilde{c}_I A_3 + a_3 A_1 A_2, \end{aligned}\right\} \tag{1.2}$$

where terms of third and higher order in the wave amplitudes are neglected. Here, * denotes a complex conjugate, the two-dimensional wave is that with amplitude A_3, and $\alpha\tilde{c}_I$ and $\frac{1}{2}\alpha c_I$ are the linear growth or decay rates of the two-dimensional and oblique waves respectively. There are three (usually complex) second-order interaction coefficients a_i $(i = 1, 2, 3)$ but, because of symmetry, $a_1 = a_2$. Asymptotic analysis for large R (in I, § 4) reveals that $|a_1|$ and $|a_2|$ are surprisingly large, being $O(R)$ while $|a_3|$ remains $O(1)$. This implies that a mechanism is available for preferential amplification of the oblique waves. It is also of interest that when $c_I = \tilde{c}_I = 0$ equations (1.2) have a particular solution (given in I, § 7) for which all three waves attain an infinite amplitude after a *finite* time (except for the special case $\arg a_1 + \arg a_2 = \pm\pi$, for which the total wave energy remains constant). This is reminiscent of the 'explosive instability' of plasma physics (Sagdeev & Galeev 1969) and the 'instability burst' of Hocking *et al.* (1972). Of course, the equations cease to be valid approximations before the singularity is attained, but it is at least plausible that these results signify the existence of genuine physical phenomena characterized by 'superexponential' growth of the disturbance energy.

It is clearly of value to extend the analysis for resonant triads to third order in the wave amplitudes in order to permit a proper comparison with the non-resonant case. The third-order equations must have the form

$$\left.\begin{aligned} dA_1/dt &= \tfrac{1}{2}\alpha c_I A_1 + a_1 A_3 A_2^* + A_1(a_{11}|A_1|^2 + a_{12}|A_2|^2 + a_{13}|A_3|^2), \\ dA_2/dt &= \tfrac{1}{2}\alpha c_I A_2 + a_2 A_3 A_1^* + A_2(a_{21}|A_1|^2 + a_{22}|A_2|^2 + a_{23}|A_3|^2), \\ dA_3/dt &= \alpha \tilde{c}_I A_3 + a_3 A_1 A_2 + A_3(a_{31}|A_1|^2 + a_{32}|A_2|^2 + a_{33}|A_3|^2), \end{aligned}\right\} \tag{1.3}$$

and it may be inferred from symmetry that

$$a_{11} = a_{22}, \quad a_{13} = a_{23}, \quad a_{12} = a_{21}, \quad a_{31} = a_{32}.$$

The object of the analysis is therefore to determine the five third-order interaction coefficients a_{11}, a_{12}, a_{13}, a_{31} and a_{33}. We note that a_{33} is just the Landau constant λ for the two-dimensional wave A_3. It is of special interest to discover whether, for large R, these interaction coefficients have large magnitudes (proportional to some positive power of R) like the second-order coefficients a_1 and a_2.

Such information furnishes greater insight into the nature of the nonlinear mechanisms of instability and indicates the probable domain of validity of the perturbation analysis for large R.

2. Formulation of the problem

The physical situation is as in II, to which the reader is referred for fuller details. All variables are dimensionless relative to characteristic length and velocity scales L and V and the constant fluid density ρ. The primary velocity profile

$$[u, v, w] = [\bar{u}(x_3), 0, 0] \quad (0 \leqslant x_3 \leqslant l)$$

confined between plane boundaries at $x_3 = 0, l$ (where l may be taken as unity for channel flows and infinity for boundary layers) is perturbed by three waves with x_1, x_2, t periodicities of the forms $\exp i\theta_j$ ($j = 1, 2, 3$) respectively, where

$$\theta_1 = \tfrac{1}{2}\alpha x_1 + \beta x_2 - \tfrac{1}{2}\alpha c_R t, \quad \theta_2 = \tfrac{1}{2}\alpha x_1 - \beta x_2 - \tfrac{1}{2}\alpha c_R t, \quad \theta_3 = \alpha x_1 - \alpha c_R t,$$

α, β and c_R being real constants. Since $\theta_1 + \theta_2 = \theta_3$ these waves comprise a resonant triad, and we here assume that the primary flow is such that a triad of this kind does indeed exist. The boundary conditions for the problem will normally be

$$u - \bar{u}(x_3) = v = w = 0 \quad (x_3 = 0, l). \tag{2.1}$$

For reasons outlined in II, § 6 we here adopt a more usual 'direct' analysis based on the Navier–Stokes equations rather than the variational method of II. We denote by A a (dimensionless) number characteristic of the wave amplitudes and by $O(A^3)$ those terms of third or higher order in the wave amplitudes. We then write the Cartesian velocity components (u, v, w) and the pressure p of the perturbed flow as

$$u = \bar{u} + \bar{\bar{u}} + \mathrm{Re}\left\{\sum_{j=1}^{3} (u_j \exp i\theta_j + u_{jj} \exp 2i\theta_j) + \sum_{j=1}^{2} u_{j3} \exp i(\theta_j + \theta_3) + u_{1\,-2} \exp i(\theta_1 - \theta_2)\right\} + O(A^3), \tag{2.2a}$$

$$v = \bar{\bar{v}} + \mathrm{Re}\left\{\sum_{j=1}^{3} v_j \exp i\theta_j + \sum_{j=1}^{2} (v_{jj} \exp 2i\theta_j + v_{j3} \exp i(\theta_j + \theta_3)) + v_{1\,-2} \exp i(\theta_1 - \theta_2)\right\} + O(A^3), \tag{2.2b}$$

$$w = \mathrm{Re}\left\{\sum_{j=1}^{3} (w_j \exp i\theta_j + w_{jj} \exp 2i\theta_j) + \sum_{j=1}^{2} w_{j3} \exp i(\theta_j + \theta_3) + w_{1\,-2} \exp i(\theta_1 - \theta_2)\right\} + O(A^3), \tag{2.2c}$$

$$p = x_1 p_I + \bar{\bar{p}} + \mathrm{Re}\left\{\sum_{j=1}^{3} (p_j \exp i\theta_j + p_{jj} \exp 2i\theta_j) + \sum_{j=1}^{2} p_{j3} \exp i(\theta_j + \theta_3) + p_{1\,-2} \exp i(\theta_1 - \theta_2)\right\} + O(A^3). \tag{2.2d}$$

The terms $[u_j, v_j, w_j, p_j] \exp i\theta_j$ represent the three waves, p_I is the imposed longitudinal pressure gradient, $\bar{\bar{u}}$, $\bar{\bar{v}}$ and $\bar{\bar{p}}$ are modifications to the mean velocity and pressure owing to the nonlinear Reynolds stresses and the remaining terms represent the second-order (sum and difference) harmonics. We shall omit all $O(A^3)$

terms with exponents *not* equal to one of the $i\theta_j$, but retain $O(A^3)$ terms with these periodicities.

Because of the directional properties of the various wavelike components it is best to write

$$\left.\begin{aligned} u_1 &= \gamma^{-1}(\tfrac{1}{2}\alpha\hat{u}_1-\beta\hat{v}_1), & v_1 &= \gamma^{-1}(\beta\hat{u}_1+\tfrac{1}{2}\alpha\hat{v}_1),\\ u_{11} &= \gamma^{-1}(\tfrac{1}{2}\alpha\hat{u}_{11}-\beta\hat{v}_{11}), & v_{11} &= \gamma^{-1}(\beta\hat{u}_{11}+\tfrac{1}{2}\alpha\hat{v}_{11}),\\ u_{13} &= \gamma_0^{-1}(\tfrac{3}{2}\alpha\hat{u}_{13}-\beta\hat{v}_{13}), & v_{13} &= \gamma_0^{-1}(\beta\hat{u}_{13}+\tfrac{3}{2}\alpha\hat{v}_{13}),\end{aligned}\right\} \tag{2.3}$$

where
$$\gamma \equiv (\tfrac{1}{4}\alpha^2+\beta^2)^{\frac{1}{2}}, \quad \gamma_0 \equiv (\tfrac{9}{4}\alpha^2+\beta^2)^{\frac{1}{2}};$$

then the quantities with a caret represent the velocity components perpendicular and parallel to the respective 'wave crests'. Corresponding transformations for the components u_2, v_2, u_{22}, v_{22}, u_{23} and v_{23} are obtained on replacing β by $-\beta$ above. We then introduce series expansions in powers of the (small but finite) complex wave amplitudes $A_j(t)$ ($j = 1,2,3$) of the members of the resonant triad, namely

$$\bar{\bar{u}} = \sum_{j=1}^{3} |A_j|^2 f_j + O(A^3), \qquad \bar{\bar{v}} = \sum_{j=1}^{2} |A_j|^2 h_j + O(A^3), \tag{2.4 a,b}$$

$$p_I = p^0 + \sum_{j=1}^{3} |A_j|^2 p_{Ij} + O(A^3), \quad \bar{\bar{p}} = \sum_{j=1}^{3} |A_j|^2 p_j + O(A^3), \tag{2.4 c,d}$$

$$w_k = A_k\chi_k^{(1)} + A_3A_{3-k}^*\chi_k^{(2)} + A_k\sum_{j=1}^{3}|A_j|^2\chi_k^{(2+j)} + O(A^4), \tag{2.4 e}$$

$$w_3 = A_3\chi_3^{(1)} + A_1A_2\chi_3^{(2)} + A_3\sum_{j=1}^{3}|A_j|^2\chi_3^{(2+j)} + O(A^4), \tag{2.4 f}$$

$$\hat{v}_k = A_k\psi_k^{(1)} + A_3A_{3-k}^*\psi_k^{(2)} + O(A^3), \quad v_3 = A_1A_2\psi_3^{(2)} + O(A^3), \tag{2.4 g,h}$$

$$w_{jj} = A_j^2\chi_{jj}^{(2)} + O(A^3), \qquad w_{k3} = A_kA_3\chi_{k3}^{(2)} + O(A^3), \tag{2.4 i,j}$$

$$\hat{v}_{kk} = A_k^2\psi_{kk}^{(2)} + O(A^3), \quad \hat{v}_{k3} = A_kA_3\psi_{k3}^{(2)} + O(A^3), \tag{2.4 k,l}$$

$$w_{1-2} = A_1A_2^*\chi_{1-2}^{(2)} + O(A^3), \quad u_{1-2} = A_1A_2^*\phi_{1-2}^{(2)} + O(A^3), \tag{2.4 m,n}$$

where i and j take the values 1, 2 and 3 and k takes the values 1 and 2. Here, p^0 and p_{Ii} are constants and all other symbols introduced denote functions of x_3 only. Corresponding expansions for the remaining variables $\hat{u}_k$, u_3, $\hat{u}_{kj}$, u_{33} and v_{1-2} are found from the continuity relations

$$\left.\begin{aligned} i\gamma\hat{u}_k + Dw_k = 0, \quad i\alpha u_3 + Dw_3 = 0, \quad 2i\alpha u_{33} + Dw_{33} = 0,\\ 2i\gamma\hat{u}_{kk} + Dw_{kk} = 0, \quad 2i\gamma_0\hat{u}_{k3} + Dw_{k3} = 0, \quad 2i\beta v_{1-2} + Dw_{1-2} = 0,\end{aligned}\right\} \tag{2.5}$$

where $D \equiv \partial/\partial x_3$. Since the governing equations were formed from the Navier–Stokes equations by elimination of the pressure terms, we do not require similar expansions for the components p_j, p_{jk} and p_{1-2}. The boundary conditions for the various functions are readily inferred from (2.1). Note that the suffixes do *not* denote Cartesian tensor indices and that no summation over repeated indices is implied.

Henceforth we assume that the imposed longitudinal pressure gradient remains constant, so that $p_{Ii} = 0$ (there seems little to choose here between this and the alternative assumption of constant volume flux). We also assume that no second-order spanwise pressure gradient can occur. In addition, we expand the time derivatives of $A_i(t)$ in the form given by (1.3) involving the linear growth or decay rates $\frac{1}{2}\alpha c_I$ and $\alpha\tilde{c}_I$ and the constant interaction coefficients a_i and a_{ij}.

Substitution into the Navier–Stokes momentum equations, linearization in the $A_i(t)$ and elimination of pressure terms by cross-differentiation yields the equations of linear theory, namely [II, equations (5.4)]

$$\left.\begin{aligned} L_1[\chi_k^{(1)}]+\tfrac{1}{2}\alpha c_I L_4[\chi_k^{(1)}] &= 0 \\ G_k-\tfrac{1}{2}\alpha c_I \psi_k^{(1)} &= 0 \end{aligned}\right\} \quad (k=1,2), \tag{2.6 a, b}$$

$$L_3[\chi_3^{(1)}]+\alpha\tilde{c}_I L_5[\chi_3^{(1)}] = 0, \tag{2.6 c}$$

$$L_1[\chi_k^{(1)}] \equiv \tfrac{1}{2}i\alpha[(\bar{u}-c_R)(D^2-\gamma^2)-D^2\bar{u}]\chi_k^{(1)}-R^{-1}(D^2-\gamma^2)^2\chi_k^{(1)}, \tag{2.7 a}$$

$$L_3[\chi_3^{(1)}] \equiv i\alpha[(\bar{u}-c_R)(D^2-\alpha^2)-D^2\bar{u}]\chi_3^{(1)}-R^{-1}(D^2-\alpha^2)^2\chi_3^{(1)}, \tag{2.7 b}$$

$$G_k \equiv R^{-1}(D^2-\gamma^2)\psi_k^{(1)}-\tfrac{1}{2}i\alpha(\bar{u}-c_R)\psi_k^{(1)}+(-1)^{k+1}\beta\gamma^{-1}D\bar{u}\chi_k^{(1)}, \tag{2.7 c}$$

$$L_4[\chi_k^{(1)}] \equiv (D^2-\gamma^2)\chi_k^{(1)}, \quad L_5[\chi_3^{(1)}] \equiv (D^2-\alpha^2)\chi_3^{(1)}, \tag{2.7 d, e}$$

and the boundary conditions at $x_3 = 0, l$ are

$$\chi_i^{(1)} = D\chi_i^{(1)} = \psi_k^{(1)} = 0 \quad (i=1,2,3;\ k=1,2;\ x_3=0,l). \tag{2.8}$$

The resultant eigenvalue problems determine the complex phase velocities c_R+ic_I and $c_R+i\tilde{c}_I$, the *real* parts of which must be equal in order to satisfy the resonance condition. We may normalize these solutions such that

$$\chi_1^{(1)} = \chi_2^{(1)}, \quad \psi_1^{(1)} = -\psi_2^{(1)}. \tag{2.9}$$

3. Nonlinear analysis

If the second-order terms containing $A_3A_2^*$, $A_3A_1^*$ and A_1A_2 are retained, we recover the second-order equations

$$L_1[\chi_k^{(2)}]+(\tfrac{1}{2}\alpha c_I+\alpha\tilde{c}_I)L_4[\chi_k^{(2)}] = F_1-a_k L_4[\chi_1^{(1)}] \quad (k=1,2), \tag{3.1 a}$$

$$L_3[\chi_3^{(2)}]+\alpha c_I L_5[\chi_3^{(2)}] = F_3-a_3 L_5[\chi_3^{(1)}], \tag{3.1 b}$$

$$\begin{aligned} F_1 \equiv \tfrac{1}{4}\{&(\alpha^2\gamma^{-2}-2)\chi_3^{(1)}(D^2-\gamma^2)D\chi_1^{(1)*}+(\alpha^2\gamma^{-2}-3)D\chi_3^{(1)}(D^2-\gamma^2)\chi_1^{(1)*} \\ &-2D\chi_1^{(1)*}(D^2-\alpha^2)\chi_3^{(1)}-\chi_1^{(1)*}(D^2-\alpha^2)D\chi_3^{(1)} \\ &-2i\alpha\beta\gamma^{-1}(\chi_3^{(1)}D^2\psi_1^{(1)*}+D\chi_3^{(1)}D\psi_1^{(1)*}+\gamma^2\chi_3^{(1)}\psi_1^{(1)*})\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F_3 \equiv -\tfrac{1}{2}\alpha^2\gamma^{-2}\{&D[\chi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)}]-(\alpha^2\gamma^{-2}-2)D\chi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)} \\ &-2i\alpha\beta\gamma^{-1}[\psi_1^{(1)}(D^2-\gamma^2)\chi_1^{(1)}+D\psi_1^{(1)}D\chi_1^{(1)}-\gamma^2\alpha^{-2}D^2(\chi_1^{(1)}\psi_1^{(1)})] \\ &+4\beta^2\psi_1^{(1)}D\psi_1^{(1)}\}, \end{aligned} \tag{3.3}$$

which are readily shown to be identical with results II (5.11) and II (5.3) on noting result (2.9) and making minor changes in notation. It follows that, as shown in II, the appropriate values of the second-order interaction coefficients a_j are

$$a_1 = a_2 = \left(\int_0^l \Psi_1^{(1)} F_1\,dx_3\right)\Big/\left(\int_0^l \Psi_1^{(1)} L_4[\chi_1^{(1)}]\,dx_3\right), \tag{3.4}$$

$$a_3 = \left(\int_0^l \Psi_3^{(1)} F_3\,dx_3\right)\Big/\left(\int_0^l \Psi_3^{(1)} L_5[\chi_3^{(1)}]\,dx_3\right), \tag{3.5}$$

where $\Psi_1^{(1)}$ and $\Psi_3^{(1)}$ are the solutions of the linear equations adjoint to (2.6 a, c) with homogeneous boundary conditions corresponding to (2.8).

In addition, result (2.9) ensures that the function $\psi_3^{(2)}$ satisfies a *homogeneous* equation, and this has only the trivial solution $\psi_3^{(2)} = 0$. Consequently, all terms containing $\psi_3^{(2)}$ may be omitted from the subsequent analysis.

On retaining third-order terms and equating to zero the coefficients of the $A_i|A_j|^2$ $(i, j = 1, 2, 3)$, the following third-order equations are ultimately recovered for the functions $\chi_i^{(2+j)}$:

$$\left.\begin{array}{l} L_1[\chi_k^{(2+l)}] + \tfrac{3}{2}\alpha c_I L_4[\chi_k^{(2+l)}] = F_{kl} - a_{kl} L_4[\chi_k^{(1)}] \qquad (3.6) \\ L_1[\chi_k^{(5)}] + (\tfrac{1}{2}\alpha c_I + 2\alpha \tilde{c}_I) L_4[\chi_k^{(5)}] = F_{k3} - a_{k3} L_4[\chi_k^{(1)}] \qquad (3.7) \\ L_3[\chi_3^{(2+l)}] + (\alpha c_I + \alpha \tilde{c}_I) L_5[\chi_3^{(2+l)}] = F_{3l} - a_{3l} L_5[\chi_3^{(1)}] \qquad (3.8) \end{array}\right\} \quad (k, l = 1, 2),$$

$$L_3[\chi_3^{(5)}] + 3\alpha \tilde{c}_I L_5[\chi_3^{(5)}] = F_{33} - a_{33} L_5[\chi_3^{(1)}], \tag{3.9}$$

where the operators $L_i[\;]$ are those defined above.

The F_{ij} are lengthy expressions. In form, they are mainly linear combinations of products of first- and second-order functions (i.e. those functions listed in table 1 and their complex conjugates) but those F_{ij} with $i \neq j$ have further terms proportional to the second-order interaction coefficients a_i $(i = 1, 2, 3)$. Because the problem is symmetric with respect to the suffixes 1 and 2, it turns out that

$$F_{21} = F_{12}, \quad F_{22} = F_{11}, \quad F_{23} = F_{13}, \quad F_{32} = F_{31} \tag{3.10}$$

(leaving only five independent functions F_{ij}), provided that the first-order solutions are normalized such that $\chi_1^{(1)} = \chi_2^{(1)}$ and $\psi_1^{(1)} = -\psi_2^{(1)}$.

For brevity, statements of all these functions are omitted;† we only give F_{33}, the simplest of the F_{ij}:

$$\begin{aligned} F_{33} \equiv & -i\alpha f_3 (D^2 - \alpha^2)\chi_3^{(1)} + i\alpha \chi_3^{(1)} D^2 f_3 - \tfrac{1}{4}\chi_3^{(1)*}(D^2 - 3\alpha^2) D\chi_{33}^{(2)} \\ & - \tfrac{1}{2} D\chi_3^{(1)*}(D^2 - 3\alpha^2)\chi_{33}^{(2)} + \tfrac{1}{4} D\chi_{33}^{(2)} D^2 \chi_3^{(1)*} + \tfrac{1}{2}\chi_{33}^{(2)} D^3 \chi_3^{(1)*}. \end{aligned} \tag{3.11}$$

By an argument similar to that of II, the appropriate values of the third-order interaction coefficients are found to be

$$\left.\begin{array}{l} a_{ki} = \left(\displaystyle\int_0^l \Psi_k^{(1)} F_{ki}\, dx_3\right) \Big/ \left(\displaystyle\int_0^l \Psi_k^{(1)} L_4[\chi_k^{(1)}]\, dx_3\right) \\ a_{3i} = \left(\displaystyle\int_0^l \Psi_3^{(1)} F_{3i}\, dx_3\right) \Big/ \left(\displaystyle\int_0^l \Psi_3^{(1)} L_5[\chi_3^{(1)}]\, dx_3\right) \end{array}\right\} \quad (k = 1, 2;\; i = 1, 2, 3). \tag{3.12}$$

In order that the interaction coefficients a_i and a_{ij} and the wave amplitudes A_i are uniquely defined, it is necessary to specify the normalizations imposed on the functions $\chi_i^{(1)}$. Reynolds & Potter (1967) used the normalizations $\chi_3^{(1)} = -i$ and $D\chi_3^{(1)} = -i$ at the centre-line of the channel (i.e. at $x_3 = 1$ with $l = 2$) for the even and odd modes of plane Poiseuille flow. On the other hand, Pekeris & Shkoller (1967) chose $\chi_3^{(1)} = -i\alpha$ and $D\chi_3^{(1)} = -i\alpha$ at $x_3 = 1$ for these modes. For Blasius flow, computations described in the appendix to this paper by Dr F. Hendriks employ the normalization $\chi_j^{(1)} = 1$ at $x_3 = 5$ $(j = 1, 2, 3)$.

† Copies of a supplement containing these may be obtained from the authors or the JFM Editorial Office, DAMTP, Silver Street, Cambridge CB3 9EW.

For the purposes of this paper, a specific choice of normalization need not be made. We insist only that this choice ensures that, over most of the flow domain (but excluding the critical layer and viscous wall layer), the functions $\chi_j^{(1)}$ and $D\chi_j^{(1)}$ $(j = 1, 2, 3)$ remain $O(1)$ in magnitude as $R \to \infty$. This condition is met, for example, by the particular normalizations mentioned above.

Clearly, the normalizations employed for the adjoint functions $\Psi_i^{(1)}$ do not affect the values of a_i and a_{ij}; but we note that expressions (3.4), (3.5) and (3.12) are simplified if we impose the normalizations

$$\int_0^l \Psi_k^{(1)} L_4[\chi_k^{(1)}]\,dx_3 = \int_0^l \Psi_3^{(1)} L_5[\chi_3^{(1)}]\,dx_3 = 1 \quad (k = 1, 2).$$

We also note that interchanging the suffixes 1 and 2 does not alter the values of the a_{ij}. Unlike results (3.4) and (3.5), where the integrals depend only on the linear solutions, expressions (3.12) cannot be evaluated immediately (even in principle!) since they involve several second-order quantities yet to be determined. The governing equations for these second-order quantities are derived by extracting from the equations of motion the appropriate terms quadratic in $A_j(t)$ and their complex conjugates. These are as follows.

The second-order modifications to the mean flow, designated by f_j and f_k, satisfy

$$\left.\begin{aligned} &\alpha c_I f_k - R^{-1}D^2 f_k \\ &\qquad = \tfrac{1}{2}\operatorname{Re}\{\beta\gamma^{-1}D(\psi_1^{(1)*}\chi_1^{(1)}) + \tfrac{1}{2}i\alpha\gamma^{-2}\chi_1^{(1)}D^2\chi_1^{(1)*}\} \\ &\alpha c_I h_k - R^{-1}D^2 h_k \\ &\qquad = \tfrac{1}{2}(-1)^k\operatorname{Re}\{\tfrac{1}{2}\alpha\gamma^{-1}D(\psi_1^{(1)*}\chi_1^{(1)}) - i\beta\gamma^{-2}\chi_1^{(1)}D^2\chi_1^{(1)*}\} \end{aligned}\right\} \quad (k = 1, 2), \quad \begin{aligned}&(3.13a)\\&(3.13b)\end{aligned}$$

$$2\alpha\tilde{c}_I f_3 - R^{-1}D^2 f_3 = \tfrac{1}{2}\operatorname{Re}\{-i\alpha^{-1}\chi_3^{(1)*}D^2\chi_3^{(1)}\}, \tag{3.13c}$$

while the timewise-aperiodic but spanwise-varying terms, designated by $\chi_{1-2}^{(2)}$ and $\phi_{1-2}^{(2)}$, satisfy

$$\begin{aligned} &\alpha c_I(D^2 - 4\beta^2)\chi_{1-2}^{(2)} - R^{-1}(D^2 - 4\beta^2)^2\chi_{1-2}^{(2)} \\ &\quad = \beta^2\gamma^{-4}D[4\beta^2|D\chi_1^{(1)}|^2 + \alpha^2\gamma^2(|\chi_1^{(1)}|^2 + |\psi_1^{(1)}|^2) - \gamma^2 D^2(|\chi_1^{(1)}|^2)] \\ &\qquad + \operatorname{Im}\{\alpha\beta\gamma^{-3}D[(3\beta^2 - \tfrac{1}{4}\alpha^2)\psi_1^{(1)}D\chi_1^{(1)*} - \gamma^2 D\psi_1^{(1)}\chi_1^{(1)*}] - 4\alpha\beta^3\gamma^{-1}\psi_1^{(1)}\chi_1^{(1)*}\}, \end{aligned} \tag{3.14a}$$

$$\begin{aligned} &\alpha c_I\phi_{1-2}^{(2)} - R^{-1}(D^2 - 4\beta^2)\phi_{1-2}^{(2)} + \chi_{1-2}^{(2)}D\bar{u} \\ &\quad = \beta\gamma^{-1}\operatorname{Re}\{-\psi_1^{(1)}D\chi_1^{(1)*} + D\psi_1^{(1)}\chi_1^{(1)*}\} + \tfrac{1}{2}\alpha\gamma^{-2}\operatorname{Re}\{i\chi_1^{(1)}D^2\chi_1^{(1)*}\}. \end{aligned} \tag{3.14b}$$

We note that the latter terms represent a spanwise-periodic 'longitudinal-vortex' distortion of the primary flow, like that studied by Benney & Lin (1960) and Benney (1961, 1964). The results $\chi_1^{(1)} = \chi_2^{(1)}$ and $\psi_1^{(1)} = -\psi_2^{(1)}$ have been used to simplify the right-hand sides.

The second-order modification $\psi_1^{(2)}$ to the wave of periodicity $i\theta_1$ satisfies

$$\begin{aligned} &[\tfrac{1}{2}i\alpha(\bar{u} - c_R) + \tfrac{1}{2}\alpha c_I + \alpha\tilde{c}_I]\psi_1^{(2)} - R^{-1}(D^2 - \gamma^2)\psi_1^{(2)} - \beta\gamma^{-1}D\bar{u}\chi_1^{(2)} \\ &\quad = -a_1\psi_1^{(1)} + \tfrac{1}{4}\psi_1^{(1)*}D\chi_3^{(1)} - \tfrac{1}{2}\gamma^{-2}(\tfrac{1}{4}\alpha^2 - \beta^2)\chi_3^{(1)}D\psi_1^{(1)*} \\ &\qquad - \tfrac{1}{2}i\beta\alpha^{-1}\gamma^{-3}(\gamma^2\chi_1^{(1)*}D^2\chi_3^{(1)} - \alpha^2\chi_3^{(1)}D^2\chi_1^{(1)*}). \end{aligned} \tag{3.15}$$

This is the component parallel to the wave crests, and is related to the perpendicular component $\chi_1^{(2)}$, which satisfies (3.1a). Note the appearance of the second-order interaction coefficient a_1 on the right-hand side. The corresponding equation for $\psi_2^{(2)}$ confirms that $\psi_2^{(2)} = -\psi_1^{(2)}$, as might be expected.

The equations for the $\chi_i^{(2)}$ are given by (3.1), (3.4) and (3.5). The equations for the remaining nine second-order functions of the type $\chi_{ij}^{(2)}$ or $\psi_{kj}^{(2)}$ are omitted for brevity but are stated in Usher (1974) (closely related equations are also given by Stuart 1962, § 3). The form of the equations for the $\chi_{ij}^{(2)}$ is somewhat similar to that of (3.1), and that of the equations for $\psi_{kj}^{(2)}$ resembles (3.15). The right-hand sides are known from linear theory, but differ from (3.1) and (3.15) in that no linear terms (in a_j) are present and that no complex-conjugate functions appear. These equations are stated in full in the unpublished supplement available from the Editorial Office and authors.

4. Asymptotic theory

In order to make further progress, one may employ numerical computation to evaluate the interaction coefficients for particular cases, as was done by Reynolds & Potter (1967) and Pekeris & Shkoller (1967, 1969); effectively, they computed a_{33} for Poiseuille and Couette–Poiseuille flow. Alternatively, one may seek further simplifying assumptions which, at the expense of precise accuracy, enable the general analysis to be taken further. We adopt the latter approach in developing an asymptotic theory which, under well-defined conditions, yields valid approximations for sufficiently large values of the Reynolds number R. Such a development constitutes a logical extension to nonlinear stability theory of the asymptotic analysis so effectively applied to linear theory, notably by Lin (1955) and Reid (1965).

The asymptotic analysis for the second-order problem was constructed in I, where explicit asymptotic approximations were given for the second-order coefficients a_j (I, §§ 4, 5). It turns out that $|a_1|$, $|a_2| = O(R)$ while $|a_3|$ remains $O(1)$ as $R \to \infty$. The $O(R)$ magnitude of a_1 and a_2 arises because the integral of $\Psi_1^{(1)} F_1$ in (3.4) is dominated by a large contribution from the vicinity of the 'critical layer', where the flow velocity $\bar{u}(x_3)$ nearly equals the downstream propagation velocity c_R of the waves. When there is more than one such layer, as in Poiseuille flow, the contributions from both layers must of course be retained. In I, it was assumed that there is a single critical layer and that the velocity profile $\bar{u}(x_3)$ is of boundary-layer type. Here, for simplicity, we discuss the contribution from a single critical layer, but extension of the analysis to more than one layer is immediate.

To understand the role of the critical layer, it is first necessary to consider the *inviscid* approximations for the solutions of linear theory. The inviscid estimates for the various functions $\Psi_j^{(1)}$, $D\chi_j^{(1)}$ and $\psi_k^{(1)}$ $(j = 1, 2, 3;\ k = 1, 2)$ are normally singular at one of the critical points in the complex x_3 plane, where $\bar{u}(x_3)$ equals the complex phase velocity $c_R + ic_I$ (for the oblique waves) or $c_R + i\tilde{c}_I$ (for the two-dimensional wave). Since $|c_I|$ and $|\tilde{c}_I|$ are usually very much smaller than $|c_R|$ in cases of interest, these two critical points are almost coincident and lie very close to the real x_3 axis. Explicitly, we assume that

$$|c_I|,\ |\tilde{c}_I| \ll (D\bar{u}_c)^{\frac{2}{3}}(\alpha R)^{-\frac{1}{3}} \quad (R \to \infty) \tag{4.1}$$

(see Lin 1955, chap. 8). Denoting both critical points by x_{3c}, the inviscid estimates close to x_{3c} are (see Reid 1965)

$$\left.\begin{aligned} \chi_j^{(1)} &\sim C_j[1+(D^2\bar{u}/D\bar{u})_c(x_3-x_{3c})\log(x_3-x_{3c})]+O(|x_3-x_{3c}|),\\ \Psi_j^{(1)} &\sim C_j(D\bar{u})_c^{-1}(x_3-x_{3c})^{-1}+O(\log|x_3-x_{3c}|),\\ \psi_k^{(1)} &\sim (-1)^k\,2i\beta(\alpha\gamma)^{-1}C_k(x_3-x_{3c})^{-1}+O(\log|x_3-x_{3c}|), \end{aligned}\right\}\quad \begin{matrix}(x_3\to x_{3c},\\ R\to\infty),\end{matrix} \tag{4.2}$$

where the subscript c denotes evaluation at x_{3c}, the C_j are constants and we assume that $D^2\bar{u}$ and $D\bar{u}$ are non-zero at x_{3c}. We note that the singularity in $D\chi_j^{(1)}$ depends on the existence of non-zero profile curvature ($D^2\bar{u} \neq 0$) at the critical point but that Ψ_j and ψ_k are singular irrespective of the profile curvature.

Direct substitution of these estimates into the integrands $\Psi_1^{(1)}F_1$ and $\Psi_3^{(1)}F_3$ of (3.4) and (3.5) yields singularities like $(x_3-x_{3c})^{-4}$ for both if the two critical points are treated as coincident on the real axis (the supposition that these points coincide is made for heuristic reasons only and is not necessary for the subsequent analysis). However, there is a basic difference between these two integrands on account of the respective domains of validity of the inviscid approximations. Outside a small circle of radius $O[(\alpha RD\bar{u}_c)^{-\frac{1}{3}}]$ centred on x_{3c}, the full inviscid estimates for $\Psi_j^{(1)}$, $\chi_j^{(1)}$ and $\psi_k^{(1)}$ (which yield (4.2) as $x_3\to x_{3c}$) are valid asymptotic approximations as $R\to\infty$ in the sector $-\frac{7}{6}\pi \leqslant \arg(x_3-x_{3c}) \leqslant \frac{1}{6}\pi$ of the complex x_3 plane, whereas the corresponding inviscid estimates for the conjugate functions $\Psi_j^{(1)*}$, $\chi_j^{(1)*}$ and $\psi_k^{(1)*}$ are valid in the sector $-\frac{1}{6}\pi \leqslant \arg(x_3-x_{3c}) \leqslant \frac{7}{6}\pi$ (see Lin 1955, chap. 8; I, §4). Since the integrand $\Psi_3^{(1)}F_3$ and those in the denominators of (3.4) and (3.5) involve no conjugate functions, their paths of integration may be deformed to pass beneath the singularity at x_{3c}; accordingly the (possibly complex) values of these integrals remain $O(1)$ as $R\to\infty$. For $\Psi_1^{(1)}F_1$, on the other hand, the functions and their conjugates are both present and the path cannot be deformed so as to avoid the singularity and yet have a uniformly valid inviscid approximation for the integrand. Instead, viscous theory must be employed to evaluate the integrand in the vicinity of the critical layer. [Note that when c_I and $\tilde{c}_I$ are $O(1)$ and positive, as can be the case for velocity profiles with an inflexion point, an inviscid approximation remains uniformly valid for all real x_3. But the present analysis concerns *small* linear growth or decay rates satisfying (4.1), for which a viscous analysis is necessary.]

The linear viscous solutions are known as functions of the 'inner' variable

$$Z \equiv i(\tfrac{1}{2}\alpha RD\bar{u}_c)^{\frac{1}{3}}(x_3-x_{3c}) \tag{4.3}$$

and it is readily shown that the critical-layer contribution to the integral of $\Psi_1^{(1)}F_1$ is $O(R)$ (see Reid 1965; I, §4).

The derivation of asymptotic estimates for the third-order coefficients a_{ij} proceeds similarly, but is rather more complicated since it is first necessary to obtain asymptotic approximations for the various second-order functions which occur in the F_{ij}. These functions fall into two categories: those which have time-periodic components and those which do not. In the former category are the functions $\chi_j^{(2)}$, $\psi_k^{(2)}$, $\chi_{ij}^{(2)}$ and $\psi_{kj}^{(2)}$, while the latter category comprises the functions f_j, h_k, $\chi_{1-2}^{(2)}$ and $\phi_{1-2}^{(2)}$.

Functions in the first category may be treated by straightforward extension of the familiar inviscid and viscous approximations of linear theory. They are characterized by a thin viscous critical layer outside which inviscid estimates may be found (for present purposes we may ignore the thin $O[(\alpha R)^{-\frac{1}{2}}]$ viscous wall layers near $x_3 = 0$, l since it may be shown that these do not contribute significantly to the interaction integrals).

For functions in the second category, the effects of viscous diffusion are not necessarily confined to a thin critical layer. Indeed, for these functions, an 'inviscid approximation' exists only for sufficiently large growth rates $|\alpha\tilde{c}_I|$ and $|\frac{1}{2}\alpha c_I|$ [see (3.13) and (3.14)]. More precisely, in a time $O[(\alpha\tilde{c}_I)^{-1}]$, viscous diffusion penetrates a distance $O[(\alpha R\tilde{c}_I)^{-\frac{1}{2}}]$, and this is not necessarily small since the only restriction on c_I and $\tilde{c}_I$ is that introduced in (4.1). This ensures the validity of the linear viscous approximations. In particular, if $(\alpha R\tilde{c}_I)^{\frac{1}{2}} \leqslant O(l^{-1})$ viscous diffusion will be important throughout the flow domain (we may define $l = 1$ for channel flows and $l = \infty$ for boundary layers). The structure of such functions as $R \to \infty$ therefore depends on the magnitude of c_I and $\tilde{c}_I$. Among the various possibilities, it seems sensible to consider that which includes the case of neutrally stable waves: that is, we suppose that no inviscid solutions can be found, but that $(\alpha Rc_I)^{-\frac{1}{2}}$ and $(\alpha R\tilde{c}_I)^{-\frac{1}{2}}$ are both $O(1)$ or greater (this assumption may be relaxed, if required, to yield results analogous to those derived here). For instance, if $(\alpha Rc_I)^{-\frac{1}{2}}$, $(\alpha R\tilde{c}_I)^{-\frac{1}{2}} \gg 1$, the second-order mean-flow terms f_j and h_k derive from a balance between viscous diffusion and the forcing terms due to nonlinear Reynolds stresses; and acceleration terms are also absent from the 'longitudinal vortex' denoted by $\chi^{(2)}_{1-2}$ and $\phi^{(2)}_{1-2}$. However, these terms still possess a 'critical-layer' structure, since the nonlinear forcing terms themselves possess inviscid approximations outside the critical layer and viscous approximations within it.

We note that Benney (1961, 1964) chose to consider cases where $(\alpha Rc_I)^{-\frac{1}{2}}$ is *small* in his analyses of non-resonant wave interactions. This leads to solutions proportional to $(\alpha c_I)^{-1}$ for the mean-flow terms, outside a layer with thickness $O[(\alpha Rc_I)^{-\frac{1}{2}}]$ centred on x_{3c}. However, Benney overlooked the fact that such solutions are inappropriate in the critical layer (where the nonlinear forcing terms are given by their viscous approximations) since $(\alpha Rc_I)^{-\frac{1}{2}}$ must remain *greater* than the critical-layer thickness $(\alpha R)^{-\frac{1}{3}}$ in order to satisfy condition (4.1) on the validity of the viscous solutions. In fact, there should be three distinct regimes in this case: the outward inviscid regions, the inner critical layer and an intermediate layer where the forcing terms are adequately represented by inviscid theory but viscous diffusion of the mean flow must be retained.

Clearly, the calculation of asymptotic approximations for all the second-order quantities of the present problem, and the subsequent derivation from (3.12) of explicit asymptotic estimates for the interaction coefficients a_{ij}, is a lengthy and complicated task. Such calculations were performed by Usher (1974), to whose work the reader is referred for further details. However, it is a relatively simple matter to establish the *orders of magnitude* with respect to R of the various second-order components both inside and outside the critical layer. If we are content with order-of-magnitude estimates rather than explicit asymptotic approximations for the interaction coefficients a_{ij}, this limited objective is sufficient. This simplification retains the essential features of the results and is adopted here for brevity.

5. Order-of-magnitude analysis

When $(\alpha R c_I)^{-\frac{1}{3}}$, $(\alpha R \tilde{c}_I)^{-\frac{1}{3}} \geqslant O(1)$ inviscid estimates may be used to evaluate the right-hand sides of (3.13) but the diffusive terms must be retained on the left-hand sides. Since the inviscid estimates for $\Psi_j^{(1)}$ and $\psi_k^{(1)}$ are

$$\left.\begin{aligned} \Psi_k^{(1)} &= (\bar{u}-c)^{-1}\chi_k^{(1)}, \quad \Psi_3^{(1)} = (\bar{u}-\tilde{c})^{-1}\chi_3^{(1)} \\ \psi_k^{(1)} &= [(-1)^k\, 2i\beta D\bar{u}/\alpha\gamma(\bar{u}-c)]\,\chi_k^{(1)} \end{aligned}\right\} \quad (k = 1,2),$$

where $c = c_R + ic_I$ and $\tilde{c} = c_R + i\tilde{c}_I$ (cf. Reid 1965; I, §4), it may be seen that the right-hand sides are $O(c_I, \tilde{c}_I)$. This is related to the familiar result that the Reynolds stresses for neutral waves are constant except in the viscous regions. For nearly neutral waves the inviscid approximation (see Lin, chap. 8)

$$\mathrm{Re}\,\{i\chi_j^{(1)}\, D^2\chi_j^{(1)*}\} = |C_j|^2(\pi D^2\bar{u}/D\bar{u})_c\, \delta(x_3 - x_{3c}),$$

where $\delta(x)$ is Dirac's delta function, is sufficient to show that

$$|\tfrac{1}{2}\alpha f_k - (-1)^k \beta h_k|,\ |\alpha f_3| = O(R) \quad (k = 1,2)$$

outside the critical layer (we henceforth regard α, β, γ and the derivatives of $\bar{u}$ as $O(1)$; also, the linear solution is normalized such that the $|C_j|$ are $O(1)$). Inside the critical layer it is known that the linear solutions $\chi_j^{(1)}$ remain $O(1)$ but that $D^2\chi_j^{(1)}$, $\Psi_j^{(1)}$ and $\psi_k^{(1)}$ are approximated by a Lommel function, with either Z or $2^{\frac{1}{3}}Z$ as argument, times a constant $O(R^{\frac{1}{3}})$ [see I, §4, noting that the functions $\chi_3^{(1)}$, $\Psi_3^{(1)}$, $\chi_k^{(1)}$, $\Psi_k^{(1)}$ and $\psi_k^{(1)}$ $(k = 1,2)$ are denoted there by $-i\alpha\phi_3$, $-i\alpha\psi_3$, $-i\gamma\phi_k$, $-i\gamma\psi_k$ and $\hat{v}_k$ respectively]. Using these results in (3.13), it is found that to highest order in R

$$d^2(\tfrac{1}{2}\alpha f_k - (-1)^k\beta h_k)/dZ^2$$
$$= \mathrm{Re}\,\{2^{\frac{4}{3}}\alpha^{-1}|C_k|^2(\alpha R)^{\frac{2}{3}}\,[D^2\bar{u}/(D\bar{u})^{\frac{1}{3}}]_c\,[L(Z)]^*\} \quad (k = 1,2),$$

where Z as defined in (4.3) is purely imaginary and $L(Z)$ is the Lommel function. Accordingly, within the critical layer,

$$\tfrac{1}{2}\alpha f_k - (-1)^k\beta h_k \sim R^{\frac{2}{3}}M(Y) + \mathscr{A}\,Y + \mathscr{B},$$

where $Y \equiv -iZ$ is real and $\mathscr{A}$ and $\mathscr{B}$ are constants determined by matching onto the $O(R)$ outer solution. A corresponding result holds for f_3. Clearly,

$$|\tfrac{1}{2}\alpha f_k - (-1)^k\beta h_k| \quad \text{and} \quad |f_3|$$

are no greater than $O(R)$ in the critical layer, but their second derivatives with respect to x_3 are $O(R^{\frac{4}{3}})$.

The components $\beta f_k - (-1)^k\,\tfrac{1}{2}\alpha h_k$, which depend on the term $D(\psi_1^{(1)*}\chi_1^{(1)})$, may similarly be shown to satisfy a critical-layer equation of the form

$$d^2(\beta f_k - (-1)^k\,\tfrac{1}{2}\alpha h_k)/dZ^2 = O(R),$$

with corresponding solutions where

$$|\beta f_k - (-1)^k \tfrac{1}{2}\alpha h_k| = O(R), \quad |D^2(\beta f_k - (-1)^k \tfrac{1}{2}\alpha h_k)| = O(R^{\frac{5}{3}}).$$

To match onto these solutions, the inviscid solutions must have $|\beta f_k - (-1)^k \frac{1}{2}\alpha h_k|$ and their first derivatives $O(R)$. For strictly neutral waves, the second derivatives of the mean-flow terms f_j and h_j must vanish outside the critical layer. The above results establish that, in this case, the mean-flow modifications *outside* the critical layer are $O(R)$ functions which vary *linearly* with x_3. This is in accord with physical intuition, for viscous diffusion would produce just such a flow in response to a constant surface stress applied at the critical layer.

For the 'longitudinal-vortex' components $\chi^{(2)}_{1-2}$ and $\phi^{(2)}_{1-2}$ equations (3.14) reveal that

$$\chi^{(2)}_{1-2} = O(R), \quad \phi^{(2)}_{1-2} = O(R^2)$$

both outside and inside the critical layer. We note that the downstream component $\phi^{(2)}_{1-2}$ is particularly large and represents a spanwise-periodic distortion of the primary velocity profile owing to convection of momentum by the x_3 velocity component $\chi^{(2)}_{1-2}$.

For the time-periodic components it is readily verified that outside the critical layer the functions $\chi^{(2)}_3$, $\chi^{(2)}_{jj}$, $\chi^{(2)}_{k3}$, $\psi^{(2)}_{kk}$ and $\psi^{(2)}_{k3}$ $(k = 1, 2;\ j = 1, 2, 3)$ are all $O(1)$. Exceptions are $\chi^{(2)}_k$ and $\psi^{(2)}_k$ $(k = 1, 2)$, which are seen from (3.1 *a*) and (3.15) to be $O(|a_k|)$; that is, they are $O(R)$ since $|a_k|$ is $O(R)$. A viscous analysis in the critical layer (see Usher 1974 for further details) shows that

$$\left.\begin{aligned} \mathscr{D}^2\chi^{(2)}_k = O(R); \quad \mathscr{D}^2\chi^{(2)}_3, \mathscr{D}^2\chi^{(2)}_{k3} = O(R^{\frac{2}{3}}); \quad \mathscr{D}^2\chi^{(2)}_{jj} = O(R^{\frac{1}{3}}) \\ \psi^{(2)}_k = O(R^{\frac{5}{3}}); \quad \psi^{(2)}_{k3} = O(R) \qquad\qquad \end{aligned}\right\} \quad (k = 1, 2;\ j = 1, 2, 3),$$

where $\mathscr{D}^2 \equiv d^2/dZ^2$ and terms of the stated orders are functions of the inner variable Z only. In addition, lower-order terms may depend on both Z and Z^*, a fact which is sometimes significant. We note that the operator $D \equiv d/dx_3$ may usually be regarded as $O(R^{\frac{1}{3}})$ in the critical layer; but this is not so when the *regular* part of the inviscid solution remains dominant in the critical layer. The functions $\chi^{(1)}_j$ exemplify the latter situation; for in the critical layer $\chi^{(1)}_j = O(1)$ on account of the regular part C_j of the inviscid solution, but $D^2\chi^{(1)}_j$ is $O(R^{\frac{1}{3}})$ instead of $O(R^{\frac{2}{3}})$. Corresponding estimates exist for the various complex-conjugate quantities, where, in the critical layer, dependence on Z and Z^* is interchanged.

We now have order-of-magnitude estimates of all the terms occurring in the integrals in (3.12). For convenience, these are listed in table 1. These estimates enable one to determine the orders of magnitude of the interaction coefficients a_{ij}. The contributions from inside and outside the critical layer must be considered separately. Unlike the integrals for the second-order coefficients the contributions from outside the critical layer are not all $O(1)$ since several of the second-order quantities are $O(R)$ or $O(R^2)$ there. Also, some care is necessary in dealing with the critical-layer contributions. Essentially these involve integrals with respect to Z along the imaginary axis from $-i\infty$ to $+i\infty$; and when the integrals are analytic functions of Z, that is, when they do not contain functions of the conjugate

Functions $(j = 1, 2, 3; k = 1, 2)$	Orders of magnitude: Outer	Orders of magnitude: Inner
$\chi_j^{(1)}$	1	1
$D\chi_j^{(1)}$	1	$\log R$
$D^2\chi_j^{(1)}$, $\psi_k^{(1)}$, $\Psi_j^{(1)}$, $\chi_{jj}^{(2)}$	1	$R^{\frac{1}{3}}$
$D^3\chi_j^{(1)}$, $D\psi_k^{(1)}$, $D\chi_{jj}^{(2)}$, $\chi_3^{(2)}$, $\chi_{k3}^{(2)}$	1	$R^{\frac{2}{3}}$
$D^2\psi_k^{(1)}$, $D^2\chi_{jj}^{(2)}$, $D\chi_3^{(2)}$, $D\chi_{k3}^{(2)}$, $\psi_{k3}^{(2)}$	1	R
$D^3\chi_{jj}^{(2)}$, $D^2\chi_3^{(2)}$, $D^2\chi_{k3}^{(2)}$, $D\psi_{k3}^{(2)}$	1	$R^{\frac{4}{3}}$
$D^3\chi_3^{(2)}$, $D^3\chi_{k3}^{(2)}$, $D^2\psi_{k3}^{(2)}$	1	$R^{\frac{5}{3}}$
$\chi_{1-2}^{(2)}$, f_j, h_k, $[\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k]$	R	R
$D\chi_{1-2}^{(2)}$, Df_3, $D[\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k]$	R	R
$\chi_k^{(2)}$	R	$R \log R$
$D\chi_k^{(2)}$, $D^2\chi_{1-2}^{(2)}$, $D^2f_3\dagger$, Df_k, Dh_k, $D^2[\frac{1}{2}\alpha f_k + (-1)^{k+1}\beta h_k]\dagger$	R	$R^{\frac{4}{3}}$
$D^2\chi_k^{(2)}$, $\psi_k^{(2)}$, $D^3\chi_{1-2}^{(2)}$, $D^2f_k\dagger$, $D^2h_k\dagger$	R	$R^{\frac{5}{3}}$
$D^3\chi_k^{(2)}$, $D\psi_k^{(2)}$	R	R^2
$D^2\psi_k^{(2)}$	R	$R^{\frac{7}{3}}$
$\phi_{1-2}^{(2)}$, $D\phi_{1-2}^{(2)}$, $D^2\phi_{1-2}^{(2)}$	R^2	R^2

TABLE 1. Orders of magnitude of functions in 'outer' and 'inner' (critical layer) regions. Functions marked with a dagger are zero in outer regions for strictly neutral waves, but would be $O(R)$ there if the condition $p_{Ii} = 0$ and its spanwise counterpart (see §2) were not imposed.

variable Z^*, contour integration round the semicircle at infinity reveals that such contributions are precisely zero (using the fact that $L[Z] \sim Z^{-1}$ as $|Z| \to \infty$ in an appropriate sector of the complex plane, together with other similar results). In fact, such integration is tantamount to indenting *under* the singularity at x_{3c} in cases where inviscid approximations remain valid, which was discussed earlier.

When terms in the integrand contain functions of both Z and Z^* in the critical layer, or equivalently, when the domains of validity of the inviscid approximations are such that one cannot indent underneath the singularity at x_{3c}, it is best to transform to the real variable Y defined as $Z = iY$, so that the path of integration is along the real Y axis from $-\infty$ to ∞. The integrands are rather complicated functions of Y involving integrals of Lommel functions (now written in the form $L_R(Y) + iL_I(Y)$) and Airy (or modified Hankel) functions, which would require numerical integration. However, the R dependence is contained in multiplicative constants and consequently the asymptotic form of the a_{ij} may be established for large R.

First, we note that the orders of magnitude of the a_{ij} are the same as those of the integrals comprising the numerators of (3.12), since the integrals in the denominators may be evaluated by inviscid theory on indenting under the singularity at x_{3c} and are consequently $O(1)$. It is then necessary to determine the order of magnitude of each term of the F_{ij}, both inside and outside the critical layer, using the estimates given above for the various first- and second-order functions. The orders of magnitude of the dominant contributions to the integrals of the $\Psi_i^{(1)} F_{ij}$ may then be established, taking particular care with the critical-layer approximations so as not to include spuriously large contributions which in fact vanish for reasons described above. One is eventually led to the following conclusions (further details being given in Usher 1974).

The magnitudes of the interaction coefficients a_{11}, a_{22} and a_{33} are $O(R^{\frac{4}{3}})$, the dominant contributions deriving from the critical layer owing to terms of F_{kk} $(k = 1, 2)$ containing $\chi_k^{(1)} D^2(\frac{1}{2}\alpha f_k - (-1)^k \beta h_k)$ and $\chi_1^{(1)*} D^3 \chi_{11}^{(2)}$ and terms of F_{33} containing $\chi_3^{(1)} D^2 f_3$ and $\chi_3^{(1)*} D^3 \chi_{33}^{(2)}$. We note that, since these coefficients represent the third-order self-interactions of the three waves, i.e. the a_{ij} are just the respective Landau constants, they do not depend on the existence of resonance. However, the order-of-magnitude estimates do depend on the assumption that $D^2\overline{u}$ is $O(1)$ at x_{3c}; if $D^2\overline{u}_c = 0$, the orders of magnitude of the $|a_{ij}|$ will be less than $O(R^{\frac{4}{3}})$.

The coefficients a_{12} and a_{21} have magnitudes $O(R^2)$, the dominant contributions coming from *outside* the critical layer on account of terms of F_{12} (and F_{21}) which involve the $O(R^2)$ function $\phi_{1-2}^{(2)}$ (and $\phi_{1-2}^{(2)*}$) and its first two derivatives. Since these coefficients denote interactions between the two oblique waves only, these estimates too are independent of the existence of resonance. They are also insensitive to the value of $D^2\overline{u}$ at x_{3c}. It is worth noting that if, instead of two separate plane oblique waves, we had considered a single three-dimensional disturbance with x_1, x_2, t dependence of the form $2A_1(t)\cos\beta x_2 \exp[\frac{1}{2}i\alpha(x_1 - c_R t)]$, as was done, for instance, by Benney (1961, 1964) and Stuart (1962), the appropriate self-interaction terms would have been $(a_{11} + a_{12})|A_1|^2 A_1$ (found on setting $A_2 = A_1$). Accordingly, the 'Landau constant' for such a three-dimensional disturbance is $O(R^2)$ as compared with $O(R^{\frac{4}{3}})$ for a plane wave, indicating that three-dimensionality is likely to lead to stronger self-interaction than in the two-dimensional case.

For a_{13}, with analogous results for a_{23}, the integrand $F_{13}\Psi_1^{(1)}$ is dominated in the critical layer by an $O(R^3)$ term containing $a_2^* D^2 \chi_1^{(2)} \Psi_1^{(1)}$. However, this term integrates to zero since it exhibits no dependence on Z^*. The largest terms of $F_{13}\Psi_1^{(1)}$ which depend on both Z and Z^* are $O(R^{\frac{8}{3}})$ (contributed for example by a term $\chi_3^{(1)} D^2 \psi_2^{(2)*} \Psi_1^{(1)}$ and the term of second-highest order from $a_2^* D^2 \chi_1^{(2)} \Psi_1^{(1)}$), which leads to the estimate $|a_{13}| = O(R^{\frac{7}{3}})$. The dominant contribution from outside the critical layer is also large, being $O(R^2)$ on account of a term $a_2^*(D^2 - \gamma^2)\chi_1^{(2)}$ in F_{13}. We conclude that $|a_{13}|$ and $|a_{23}|$ are $O(R^{\frac{7}{3}})$, that the terms of this order derive from the critical layer, and that the result depends crucially on the existence of resonance. *Without* resonance, terms of F_{13} containing the subscript 2 must be omitted and the dominant contributions to $|a_{13}|$ and $|a_{23}|$ appear to reduce to $O(R)$.

The dominant contributions to $|a_{31}|$ and $|a_{32}|$ are $O(R^{\frac{5}{3}})$, deriving from the critical-layer estimates of several terms of F_{31} and F_{32} respectively. This result depends on resonance to the extent that the critical layers for a two-dimensional and an oblique wave coincide. When this is not the case, $|a_{31}|$ and $|a_{32}|$ are $O(R^{\frac{4}{3}})$.

We may summarize our results by re-expressing the interaction equations (1.3) for resonance as

$$\left.\begin{aligned} dA_1/dt &= \tfrac{1}{2}\alpha c_I A_1 + b_1 R A_3 A_2^* + A_1[d_1 R^{\frac{4}{3}} |A_1|^2 + d_2 R^2 |A_2|^2 + d_3 R^{\frac{7}{3}} |A_3|^2], \\ dA_2/dt &= \tfrac{1}{2}\alpha c_I A_2 + b_1 R A_3 A_1^* + A_2[d_2 R^2 |A_1|^2 + d_1 R^{\frac{4}{3}} |A_2|^2 + d_3 R^{\frac{7}{3}} |A_3|^2], \\ dA_3/dt &= \alpha \tilde{c}_I A_3 + a_3 A_1 A_2 + A_3[d_4 R^{\frac{5}{3}}(|A_1|^2 + |A_2|^2) + d_5 R^{\frac{4}{3}} |A_3|^2], \end{aligned}\right\} \quad (5.1)$$

where the (usually complex) coefficients a_3, b_1 and d_i $(i = 1, 2, \ldots, 5)$ are $O(1)$. Without resonance, but of course with the symmetric waves A_1 and A_2 coupled in phase, the corresponding equations have the form

$$\left.\begin{aligned} dA_1/dt &= \tfrac{1}{2}\alpha c_I A_1 + A_1[d_1 R^{\frac{4}{3}} |A_1|^2 + d_2 R^2 |A_2|^2 + d_3' R |A_3|^2], \\ dA_2/dt &= \tfrac{1}{2}\alpha c_I A_2 + A_2[d_2 R^2 |A_1|^2 + d_1 R^{\frac{4}{3}} |A_2|^2 + d_3' R |A_3|^2], \\ dA_3/dt &= \alpha \tilde{c}_I A_3 + A_3[d_4' R^{\frac{4}{3}}(|A_1|^2 + |A_2|^2) + d_5 R^{\frac{4}{3}} |A_3|^2], \end{aligned}\right\} \quad (5.2)$$

where d_3' and d_4' are $O(1)$ in magnitude, and similar results would apply when the downstream wavenumber of the oblique waves A_1 and A_2 is independent of that of the two-dimensional wave A_3.

Finally, we observe that, for resonant triads comprising a two-dimensional wave and two *asymmetric* plane oblique waves of differing wavenumbers and frequencies (but such that their respective exponents $i\theta_j$ satisfy $\theta_1 + \theta_2 = \theta_3$), the orders of magnitude of most of the interaction coefficients are reduced from those of (5.1). In this case, the critical layers of the three waves are distinct and it is usually possible to indent under or over the respective singularities when evaluating the integrals. The second-order coefficients a_i are then all $O(1)$ and the third-order coefficients a_{ij} are of the same magnitude as those in (5.2).

We emphasize that the above results concern only the *asymptotic form* of the interaction coefficients for large values of R, when the other parameters $\alpha, \beta, \gamma, c_R$ and the first two derivatives of $\bar{u}$ at x_{3c} are regarded as $O(1)$ quantities. Further conditions required for the validity of the perturbation analysis are discussed in the following section.

6. Discussion

First, we state the various conditions which must be satisfied for results (5.1) or (5.2) to hold. These are the explicit assumptions

(i) $\alpha, \beta, \gamma, |D\bar{u}_c|, |D^2\bar{u}_c| = O(1)$,

(ii) $\alpha R|c_I|, \alpha R|\tilde{c}_I| \leqslant O(1)$,

(iii) $R^{\frac{1}{3}} \gg 1$,

and, implicitly, the conditions

(iv) $|c_R - \bar{u}(0)|, |c_R - \bar{u}(l)| \gg R^{-\frac{1}{3}}$,

which ensure that the critical layer remains distinct from the viscous wall layers. We note that (ii) may be relaxed if required provided that (4.1) remains satisfied. A further restriction on the validity of the linear viscous approximations in the critical layer is (see Benney & Bergeron 1969; Davis 1969)

(v) $|A_j| R^{\frac{2}{3}} \ll 1 \quad (j = 1, 2, 3)$.

Conditions (i)–(v) must be satisfied for both the resonant and the non-resonant case.

Further necessary conditions may be inferred from (5.1) and (5.2) respectively for the resonant and non-resonant cases. Essentially, we require $|A_i^{-1} dA_i/dt|$ $(i = 1, 2, 3)$ to be small compared with unity in order that the amplitude modulation owing to nonlinear effects takes place on a time scale long compared with L/V, where L and V are the characteristic length and velocity scales used for non-dimensionalization (typically, L is the channel width or boundary-layer thickness and V the maximum flow velocity). For simplicity, we suppose that A_1 and A_2 are of the same order of magnitude, say $A_{1,2}$. Then, for the second-order terms of (5.1) to be sufficiently small, it is necessary that

(vi) $R|A_3|, |A_{1,2}^2/A_3| \ll 1,$

and for the third-order terms of (5.1), that

(vii) $R^2|A_{1,2}|^2, R^{\frac{7}{3}}|A_3|^2 \ll 1.$

Without resonance, the corresponding conditions from (5.2) are

(vi)′ $R^2|A_{1,2}|^2, R^{\frac{4}{3}}|A_3|^2 \ll 1.$

We observe that, strictly speaking, these conditions are necessary but not sufficient to ensure that $|A_i^{-1}dA_i/dt| \ll 1$ since the orders of magnitude with respect to R of the omitted higher-order terms of the perturbation series are unknown, and it has not been established that these series are asymptotic. We expect that no more stringent conditions are required than those above, in order to ensure that the largest nonlinear terms have been retained, but we acknowledge that some uncertainty remains.

A more formal asymptotic analysis than ours might be constructed by scaling the wave amplitudes A_j and linear growth rates $\frac{1}{2}\alpha c_I$ and $\alpha\tilde{c}_I$ as appropriate negative powers of R, introducing multiple time scales as required and finally retaining only the leading-order terms in each of the amplitude equations as $R \to \infty$. However, various different scalings are possible, and the analysis of each case of interest would probably be even more formidable than our own far-from-trivial (!) third-order analysis. In particular, if one hoped to justify retaining certain third-order terms while neglecting all fourth- and higher-order ones, the analysis would have to be pursued to at least fourth order in the wave amplitudes. From our experience of the third-order analysis, we conclude that such a venture would be ill advised.

However, we set down two illustrative examples of scaled equations. In these, the stated orders of magnitude refer *only* to those omitted terms of up to (and including) third order in the wave amplitudes. On defining $B_i \equiv R^{\frac{5}{3}}A_i$ $(i = 1, 2, 3)$ and $\tau \equiv R^{-\frac{2}{3}}t$, regarding B_i and τ as $O(1)$, equations (5.1) reduce to

$$\left.\begin{aligned} dB_1/d\tau &= b_1 B_3 B_2^* + O(R^{-\frac{1}{3}}) \\ dB_2/d\tau &= b_1 B_3 B_1^* + O(R^{-\frac{1}{3}}) \\ dB_3/d\tau &= O(\alpha\tilde{c}_I R^{\frac{2}{3}},\ R^{-1}) \end{aligned}\right\} \quad (R \to \infty),$$

where $\alpha\tilde{c}_I R^{\frac{2}{3}} \leqslant O(R^{-\frac{1}{3}})$ in virtue of condition (ii). For this case, B_3 remains constant on the time scale τ while, with appropriately chosen initial phases, both

$|B_1|$ and $|B_2|$ grow like $\exp|b_1 B_3|\tau$. The second example is given by the scaling $B_k \equiv R^{\frac{3}{2}} A_k$ $(k = 1, 2)$, $B_3 \equiv R^2 A_3$ and $\tau \equiv R^{-1}t$, which leads to

$$\left.\begin{aligned} dB_1/d\tau &= \sigma B_1 + b_1 B_3 B_2^* + d_2 B_1 |B_2|^2 + O(R^{-\frac{2}{3}}) \\ dB_2/d\tau &= \sigma B_2 + b_1 B_3 B_1^* + d_2 B_2 |B_1|^2 + O(R^{-\frac{2}{3}}) \\ dB_3/d\tau &= \tilde{\sigma} B_3 + a_3 B_1 B_2 + O(R^{-\frac{1}{3}}) \end{aligned}\right\} \quad (R \to \infty)$$

from (5.1), where $\sigma = \frac{1}{2}\alpha c_I R$ and $\tilde{\sigma} = \alpha \tilde{c}_I R$. If $|\sigma|$ and $|\tilde{\sigma}|$ are $O(1)$ the oblique waves experience first-, second- and third-order contributions of comparable magnitudes, while the two-dimensional wave is adequately described by second-order theory.

The predictions in the present paper (and in I and II) of the sizes of the various interaction coefficients can be tested to some extent by comparison with existing results which have been computed for particular flows. For the second-order coefficients a_i $(i = 1, 2, 3)$ no previously published results exist, and work is at present in hand by Professor R. E. Kelly to compute these for Blasius flow at various values of R. Some early results obtained by Dr F. Hendriks and Professor Kelly are described in the appendix, by Dr Hendriks. These are for a fixed Reynolds number (based on displacement thickness) of 882 and concern six separate symmetric resonant triads. It may be seen from table 2 that, at the higher wavenumbers, the magnitude of the coefficient a_1 $(= a_2)$ for the oblique waves is substantially larger (by a factor of about 30 for $\alpha = 0{\cdot}5$) than that of the two-dimensional coefficient a_3. This is in qualitative agreement with our results. That this is not so markedly the case at small wavenumbers is to be expected, for αR is only 88·2 for $\alpha = 0{\cdot}1$ and the conditions for validity of the asymptotic theory are not met. Indeed, for Blasius flow, the Reynolds numbers of interest are probably not large enough to encourage great confidence in asymptotic theory; nevertheless, the qualitative agreement with our results is most encouraging.

The third-order coefficients a_{ii} $(i = 1, 2, 3)$ are just the Landau constants for the respective plane waves, for which we have predicted an $O(R^{\frac{4}{3}})$ dependence. No numerical results exist for any other of the a_{ij}, but Reynolds & Potter (1967) tabulate several values of a_{ii} for Poiseuille and for Couette–Poiseuille flow. Unfortunately, only a few results are quoted for fixed α but different values of R, and no meaningful comparison can be made for Couette–Poiseuille flow. For plane Poiseuille flow, extensive tables of a_{ii} as a function of α and R are given by Pekeris & Shkoller (1967). We have displayed their data in figure 1 as curves of $|a_{ii}|$ against αR with logarithmic scales, at constant values of α. Reynolds & Potter's few points are in good agreement, but are not shown.

It may be seen that, except at small wavenumbers (when our asymptotic analysis is invalid), the value of $|a_{ii}|$ increases strongly with R. To allow comparison with our predicted $O(R^{\frac{4}{3}})$ dependence, we have superposed dashed curves with gradient $\frac{4}{3}$. There is general agreement with the gradients of the computed data. Complete agreement would be too much to expect, for $|a_{ii}|$ depends on the location of the critical layer, as well as explicitly on R, and as R changes with α fixed this location changes. However, this comparison gives a convincing demonstration of the relevance of our asymptotic analysis.

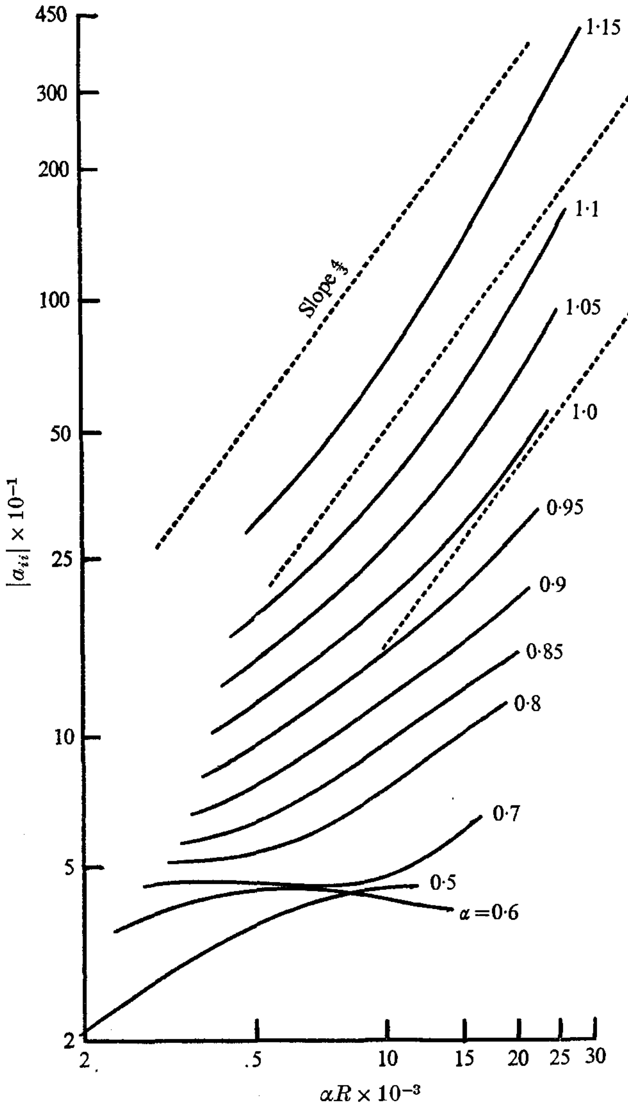

FIGURE 1. Comparison with results for plane Poiseuille flow: $|a_{ii}|$ *vs.* αR for various values of α. ——, drawn through Pekeris & Shkoller's data, with logarithmic scales; ----, gradient of $\frac{4}{3}$ predicted by asymptotic theory.

Equations (5.1) and (5.2) and the corresponding equations for asymmetric resonant triads shed considerable light on the roles of resonance and of three-dimensionality in the nonlinear stability of parallel flows. The following remarks relate to situations where conditions (i)–(vii) or (i)–(vi′) are met.

A disturbance which, at first order, is a single plane wave satisfies an equation of the form (1.1), where we have shown that $|\lambda| = O(R^{\frac{4}{3}})$ for large R. If, instead, the first-order disturbance is three-dimensional, with x_1, x_2, t dependence like $A(t)\cos\beta x_2 \exp[i\alpha(x_1 - c_R t)]$, an equation of the same form as (1.1) is again satisfied; but now $|\lambda| = O(R^2)$ for large R (this is just the special case $A_3 = 0$, $A_1 = A_2$ of (5.2)). More generally, third-order interaction coefficients $O(R^2)$ will arise whenever the first-order disturbance contains among its Fourier components a symmetric pair of (not necessarily equal) oblique plane waves.

That there are coefficients as large as $O(R^2)$ is at first sight surprising, and derives from the fact that the downstream component associated with the second-order spanwise-periodic 'longitudinal vortex' is $O(R^2)$. It is clear that three-dimensionality increases the *strength* of third-order interactions. Whether their effect is to enhance or inhibit the growth of disturbance energy is not indicated by the present analysis since estimates of the *phases* of the complex interaction coefficients are not available.

For *asymmetric* resonant triads, for which the three critical layers are distinct, the results are not particularly striking. One merely adds appropriate second-order terms with $O(1)$ interaction coefficients a_i $(i = 1, 2, 3)$ to (5.2). With *symmetric* resonant triads, on the other hand, the changes are substantial. For the oblique waves, the second-order coefficients a_1 and a_2 become $O(R)$ and the third-order coefficients a_{13} and a_{23} increase to $O(R^{\frac{7}{3}})$, essentially because of the 'superposition of critical layers'. For the two-dimensional wave, the coefficients a_{31} and a_{32} also increase, but less dramatically, from $O(R^{\frac{4}{3}})$ to $O(R^{\frac{5}{3}})$. Consequently, not only are strong second-order interactions introduced by such resonance, but the strength of the third-order interactions is also enhanced.

That there are larger interaction coefficients for oblique waves than for a two-dimensional wave, both with and without resonance, has important implications, for the possibility arises of preferential growth of the three-dimensional components. Clearly, this must sometimes occur, but firm conclusions must await detailed analyses of particular problems incorporating the phases of the interaction coefficients.

There are indications that (5.1) may sometimes play an especially important part in promoting *subcritical* instability (i.e. for $R < R_c$, the critical Reynolds number). In particular, it is likely that a three-wave resonant instability may occur with smaller initial disturbances than those necessary to yield growing (single-wave) solutions of (1.1) with $\lambda_R > 0$, and it is possible that such subcritical instability may also take place when $\lambda_R < 0$ (see Craik 1975).

It is perhaps worth noting that the very different 'kinematic-wave' analysis by Landahl (1972), of a small-scale secondary wave riding on a large-scale inhomogeneity in a shear flow, is not unconnected with the present problem. For, regarding the large-scale inhomogeneity as a long wave with wavenumber α_1 and frequency ω_1, Landahl discovers that 'focusing' of the secondary wave train takes place near the primary wave crest when the secondary-wave group velocity c_g is close to the phase velocity ω_1/α_1 of the long wave (a similar phenomenon, in an oceanographic context, was studied by Gargett & Hughes 1972). On regarding the secondary wave train not as a single component but as two components with nearly equal wavenumbers α_2 and α_3 $(\gg \alpha_1)$ and frequencies ω_2 and ω_3 $(\gg \omega_1)$, the group velocity c_g is well approximated by $c_g \equiv \partial\omega/\partial\alpha \approx (\omega_3 - \omega_2)/(\alpha_3 - \alpha_2)$. Now, if we choose the small difference $\alpha_3 - \alpha_2$ to be just α_1, the 'focusing' condition $c_g = \omega_1/\alpha_1$ yields $\omega_3 - \omega_2 = \omega_1$. That is to say, focusing occurs when the three waves form a *resonant triad* such that $\omega_3 - \omega_2 = \omega_1$ and $\alpha_3 - \alpha_2 = \alpha_1$. (We are grateful to Dr M. A. S. Ross of Edinburgh University for pointing this out to us.)

We further mention that the present work deals with situations not covered by the analyses of Hocking *et al.* (1972) and DiPrima, Eckhaus & Segel (1971). The validity of the present analysis is not restricted to the immediate locality of the critical Reynolds number R_c; instead, we require that R is large and that the linear amplification rates are sufficiently small that conditions (ii) are met. Also, while the analyses of Hocking *et al.* and DiPrima *et al.* concern the nonlinear evolution of predominantly two-dimensional disturbances (in the analysis of Hocking *et al.* the disturbance is dominated by a single plane-wave mode by the time that nonlinear effects are felt), the present analysis examines inherently three-dimensional disturbances for cases where three wave modes remain of comparable importance. The possibility of resonance among wave modes is excluded in the former studies, but is a major feature of the present work. Extension of the present analysis to incorporate spatial as well as temporal evolution of the waves, as was done by Hocking *et al.*, appears to be feasible but is not pursued here.

Our results call into question the relevance of linear estimates of the growth rates of plane waves at large R, at least in cases where those estimates satisfy condition (4.1). For, in view of the increasing strength of the nonlinear terms as R increases, the range of validity of the linear results is restricted to ever smaller wave amplitudes. At sufficiently large R, the permissible amplitudes are probably unrealistically small.

Our analysis raises similar doubts concerning the practical ranges of validity of *all* nonlinear analyses which employ expansions in ascending powers of wave amplitude. Formally, such analyses may be valid for a particular asymptotic limit: for instance, Hocking *et al.* and others use a small parameter ϵ proportional to $R - R_c$, where R_c is the critical Reynolds number, and envisage the limit $\epsilon \to 0$. But, in addition, there is a strong hope that the results of such analyses will remain good approximations for all 'reasonably small' values of the governing parameter (and hence the wave amplitudes). However, if the governing parameter does not adequately account for the Reynolds-number dependence of the nonlinear interaction coefficients, the words 'reasonably small' demand careful interpretation.

To demonstrate this point, consider the Landau-type equation

$$\dot{A} = \epsilon A + \lambda |A|^2 A,$$

where ϵ may be thought of as proportional to $R - R_c$, and R_c is a large number. In the limit $\epsilon \to 0$, we suppose that this equation is formally valid for amplitudes $|A|$ which are $O(\epsilon^{\frac{1}{2}})$. But, if the magnitude $|\lambda|$ of the Landau constant is itself large, say $O(R^n)$, and if we are in fact interested in small *finite* values of ϵ which are $O(R^{-s})$, then the equation is not necessarily a valid approximation when $|A|$ is $O(\epsilon^{\frac{1}{2}})$. Instead, the linear and second-order terms are in balance when $|A|$ is $O(R^{-\frac{1}{2}(r+s)})$, and higher-order terms must be examined to determine whether they remain acceptably small. This is not just a *mathematical* restriction on the range

α	β	γ	$\tilde{c}$	c	a_1	a_3
0·1000	0·0617	0·0794	0·2859 $-0·0461i$	0·2859 $-0·0888i$	0·5473 $+0·7013i$	0·6079 $+0·5563i$
0·2000	0·1209	0·1569	0·3394 $+0·0041i$	0·3394 $-0·0294i$	3·7350 $+1·1757i$	0·0083 $-0·2471i$
0·2540	0·1480	0·1950	0·3570 $+0·0102i$	0·3570 $-0·0122i$	6·0745 $+0·6499i$	0·3036 $-0·3394i$
0·3000	0·1705	0·2271	0·3685 $+0·0083i$	0·3685 $-0·0033i$	8·8249 $-0·1495i$	0·4305 $-0·3217i$
0·4000	0·2098	0·2891	0·3846 $-0·0107i$	0·3846 $+0·0035i$	18·8784 $-3·7073i$	0·4962 $-0·4081i$
0·5000	0·1911	0·3147	0·3834 $-0·0444i$	0·3834 $+0·0047i$	29·5892 $-6·0644i$	0·0129 $-0·9701i$

TABLE 2. Resonant triads, eigenvalues and second-order interaction coefficients for Blasius flow at $R = 882$

of validity of nonlinear analyses at large R, rather it is an indication of the increasing *physical* role of nonlinearity as R increases.

Our major qualitative conclusions may be summarized as follows.

(i) At large R, the influence of nonlinearity on the temporal evolution of wavelike disturbances is remarkably strong.

(ii) For a three-dimensional disturbance, this influence is much greater than for a two-dimensional disturbance of comparable amplitude.

(iii) Symmetric resonance at second order yields even larger nonlinear contributions.

(iv) Three-dimensionality is likely to develop very rapidly in unstable shear flows at large R.

(v) The surprising strength of the nonlinear interactions, which increases with R, limits the probable ranges of validity of linear theory and of amplitude-expansion techniques to smaller amplitudes than was previously supposed.

We wish to acknowledge helpful comments from Dr M. A. S. Ross, Professor K. Stewartson and Professor J. T. Stuart. We are particularly grateful to Professor R. E. Kelly and Dr F. Hendriks for supplying us with the results described in the appendix. One of us (J.R.U.) was supported by an S.R.C. Research Studentship while part of this work was performed.

Appendix

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To support the analysis of the mechanism of nonlinear resonant interactions, the second-order interaction coefficients a_1 and a_3 were computed for Blasius flow at $R = 882$ (based on displacement thickness and free-stream velocity). The selected triads and the corresponding values of a_1 and a_3 are given in table 2.

In order to compute interaction coefficients, the search for an oblique wave that

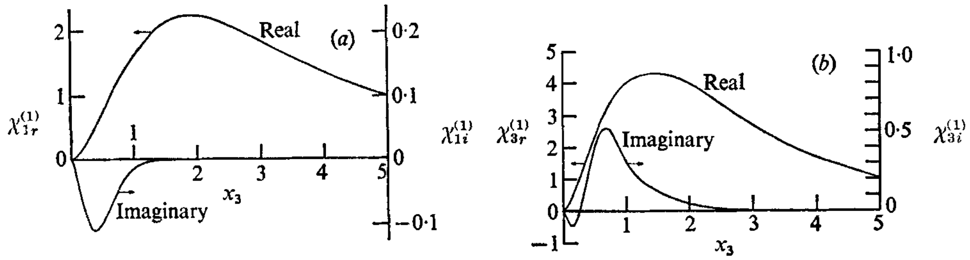

FIGURE 2. Eigenfunctions at $\alpha = 0{\cdot}5$, $R = 882$. (a) $\beta = 0{\cdot}1911$. (b) $\beta = 0$. Normalization: $\chi_k^{(1)}(5) = 1$ $(k = 1, 2)$.

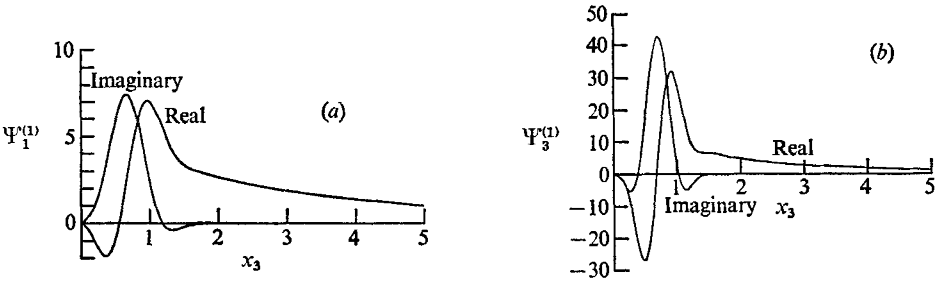

FIGURE 3. Adjoint eigenfunctions at $\alpha = 0{\cdot}5$, $R = 882$. (a) $\beta = 0{\cdot}1911$. (b) $\beta = 0$. Normalization: $\Psi_k^{(1)}(5) = 1$ $(k = 1, 2)$.

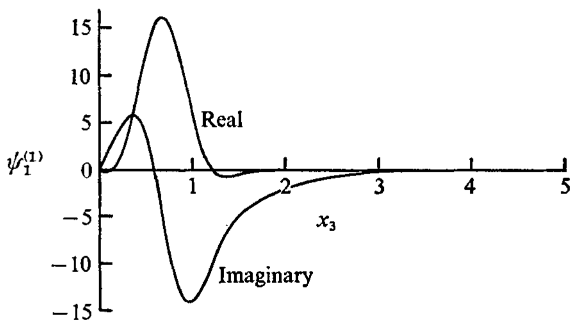

FIGURE 4. Solution $\psi_1^{(1)}$ to the cross-flow equation (2.6*b*).

will form a resonant triad with a given two-dimensional wave must be carried out first. In the present case this was done by a numerical search in the α *vs.* R stability diagram along a line of constant phase velocity until the two-dimensional counterpart (Squire's theorem) of the desired oblique wave was found. The search was speeded up by the use of Lagrange interpolation. The method used in solving the linear eigenvalue problems and their adjoints was integration with a fourth-order Runge–Kutta scheme combined with Gram–Schmidt orthonormalization after every integration step. This popular technique for treating the 'parasitic growth' problem emphasizes satisfaction of the eigenvalue criterion at the wall, while the eigenfunction need not be determined until it is called for. Figures 2 and 3 show the eigenfunctions and adjoint eigenfunctions

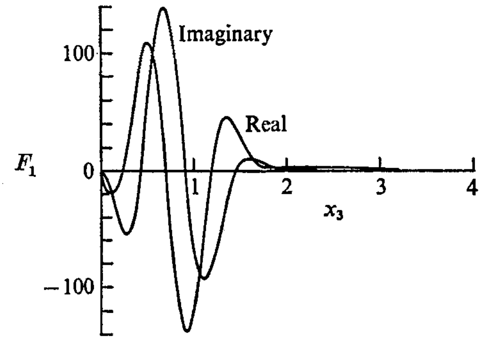

FIGURE 5. Second-order function F_1 associated with resonance in the oblique wave, defined in (3.2).

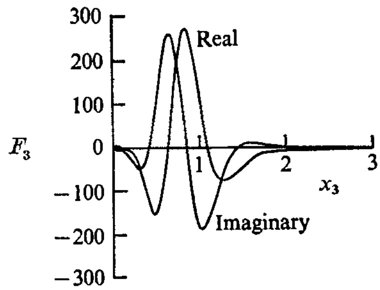

FIGURE 6. Second-order function F_3 associated with resonance of the two-dimensional wave, defined in (3.3).

of the linear stability problem for $\alpha = 0{\cdot}5$, both for the two-dimensional wave and its oblique companion. The normalization adopted is $\chi^{(1)}_k(5) = \Psi^{(1)}_k(5) = 1$ $(k = 1, 2, 3)$. The flow field in an oblique wave is three-dimensional but independent of the co-ordinate along the crest of the wave. The cross-flow follows from the solution $\psi_1^{(1)}$ to (2.6*b*). This equation is inhomogeneous and, not surprisingly, also suffers from parasitic growth problems, in this case between the homogeneous and a particular solution. Standard purification techniques apply, except for the fact that it is not possible to scale the particular solution. Instead, the norms of the homogeneous and particular solution vectors are kept identical in the usual side-by-side integration. One of the homogeneous solutions grows towards the wall and it is this function that is prevented from becoming part of the particular solution. The proper combination of the particular and homogeneous solution, the function $\psi_1^{(1)}$, is shown in figure 4. The second-order functions F_1 and F_3 in (3.2) and (3.3) are plotted in figures 5 and 6. Finally, the solvability conditions (3.4) and (3.5) yield the desired second-order interaction coefficients in table 2.

The purpose of scanning a number of triads at fixed R was to investigate whether there might exist a preferred spanwise wavenumber based on second-order effects. The data presented here do not support this suggestion sufficiently. The only point in its favour is a local maximum near $\alpha = 0{\cdot}2$ of the ratio of the magnitudes of a_1 and a_3, but it has been pointed out earlier by Craik (1971) that another candidate for a preferred β is simply the one which forms a resonant triad with the most unstable disturbance at first order.

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